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Expression of strain tensor in orthogonal curvilinear coordinates

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Abstract: Based on an analysis of connotation and extension of the concept of the orthogonal curvilinear coordinates, we have deduced a platform of strain tensor expression of Cartesian coordinates, which turns out to be a function of Lame coefficient and unit vector. By using transform matrix between Cartesian coordinates and orthogonal curvilinear coordinates, we have deduced a mathematical expression for correcting displacement vector differential in orthogonal curvilinear coordinates, and given a general expression of strain tensor in orthogonal curvilinear coordinates.

Key wards: Orthogonal Curvilinear Coordinates (OCC); Cartesian coordinates; strain tensor; transformation matrix; universal expression

1 Introduction

In geodesy and geophysics, to determine the location as well as the speed and direction of movement of a point in the crust's surface, we need to use a suitable coordinate system. Currently two coordinate systems are used in GPS observation: Cartesian (ITRF) and ellipsoidal (WGS84). The expression and derivation of crustal strain tensors at a point varies with the coordinate systems. Even in the same coordinate system, such as the Cartesian, different derivation methods were used by different authors[1-5]. For example, papers[6-7] have presented expressions of strain tensors in the spherical and the ellipsoidal coordinates, respectively. Although the expressions vary with coordinates, the volume and surface strains at a point remains same, and the strain tensors should be convertible[8] among different coordinates due to their inherent correlation. But the question is: Can we condense all of these expressions in different coordinates to a universal one? The answer is yes.

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Although the coordinate curves of different orthogonal curvilinear coordinates (OCC) are different, their partial reference frames are all rectangular. The former part shows OCC's extensive property; whereas the latter, its intensive property[9]. Just like ellipse, parabola and hyperbola can be used as mathematical expressions of conic curves of the same cone, we can also derive universally applicable strain expressions in OCC by using the expressions of strain tensor in Cartesian coordinates at a partial reference frame which represents the commonality of OCC. To our knowledge, as early as 1981 and 1992 Menahem et al[10] and Zhong Weifang et al[11], already derived universal strain expression in OCC, but in a way different from ours, which uses strain tensor expressions in Cartesian coordinates as a platform. Thus, whether in Cartesian coordinates or in OCC, the deriving methods of strain-tensor expressions are evidently not limited to one. Therefore, to find the simplest way of deriving strain-tensor expressions in OCC becomes the main topic of this paper.

2 Curvilinear coordinates and Cartesian coordinates

In an OCC system, the reference surface is a curved 2-

D surface nested in a Cartesian coordinate system. By using this reference surface, the coordinates of any point in the Euclidean space can be defined: q_1 and q_2 are the coordinates of this point's normal projection on the reference surface, q_3 is the distance between the point and the surface along the normal. The coordinate curves of q_1 and q_2 intersect perpendicularly into a 2-D network. Thus at that point a partial orthogonal frame is formed by 2 tangent vectors of the surface and the normal vector. Thus, 3 continuously differentiable and single-valued functions in Cartesian coordinates can be given in a connected domain Ω in the Euclidean space E^3

$$x_i = f_i(q_1, q_2, q_3) \quad i = 1,2,3 \tag{1}$$

$$q_1 = g_i(x_1, x_2, x_3) \quad i = 1,2,3 \tag{2}$$

In domain Ω, the curvilinear coordinate variant q_i and the Cartesian coordinates x_i are linked by a reversible, single-valued, and continuously differentiable transformation. Then the position vector $\boldsymbol{X}$ of a point in the E^3 space can be expressed as:

$$\boldsymbol{X} = (x_1, x_2, x_3) \tag{3}$$

Because (x_1, x_2, x_3) and (q_1, q_2, q_3) correspond to the same point within Ω, the Jacobian determinants of forward and reverse coordinate-transformation matrices $[\boldsymbol{T}]_{ij}$ and $[\boldsymbol{T}^{-1}]_{ij}$ are both non-zero, that is,

$$\det[\boldsymbol{T}]_{ij} = \det\left(\frac{\partial x_i}{\partial q_j}\right) \neq 0 \tag{4}$$

$$\det[\boldsymbol{T}^{-1}]_{ij} = \det\left(\frac{\partial q_i}{\partial x_j}\right) \neq 0 \tag{5}$$

In addition, the corresponding transform matrix is reversible:

$$\frac{\partial q_i}{\partial x_k}\frac{\partial x_k}{\partial q_j} = \frac{\partial q_i}{\partial q_j} = \delta_j^i \tag{6}$$

Where δ_j^i is Kronecker delta. If $i = j$, then $\delta_j^i = 1$; otherwise, $\delta_j^i = 0$.

The Jacobian determinant of transformation matrix is not equal to zero, which means that in the transformation matrix $[\boldsymbol{T}]_{ij}$ from Cartesian coordinates to curvilinear coordinates, the column vectors are not parallel to each other; similarly the row vector in the transformation matrix $[\boldsymbol{T}^{-1}]_{ij}$ from curvilinear coordinates to Cartesian coordinates are not parallel to each other either. But for the OCC system, the column vectors in $[\boldsymbol{T}]_{ij}$ are orthogonal to each other, and the row vectors in $[\boldsymbol{T}^{-1}]_{ij}$ is orthogonal to each other also. The forward and reverse Jacobian determinants are not equal to zero, which means that a particle in the surface of the earth cannot be torn into two, nor can two particles overlap into one. This is why we adopt an elastic continuum in the strain analysis.

For example, in the spherical coordinates, the Jacobian determinant $\boldsymbol{J} = \det[\boldsymbol{T}]_{ij} = r^2\sin\theta$[12], and $\boldsymbol{J} \neq 0$ only in the range $r > 0, \theta \neq 0, \pi$ domain of Ω. Like wise the Jacobian determinant of the inverse matrix $\boldsymbol{J} = \det[\boldsymbol{T}^{-1}]_{ij} = \dfrac{1}{r^2\sin\theta}$ is only non-singular within the same $r > 0, \theta \neq 0, \pi$ domain of Ω. To assure the monotropic function of both coordinates, we have to get rid of the half plane $(x_1 \geq 0, x_2 = 0)$ to assure $0 < \phi < 2\pi$. The area that should be removed is shown by the dark parts in Figure 1[13]. Figure 2[13] shows a diagram of the coordinate curves of the θ, ϕ parameters with a fixed r.

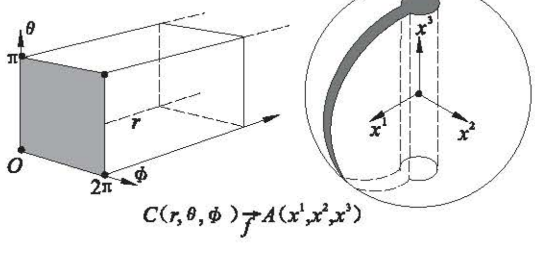

Figure 1 Jacobian nonzero non-singular region of coordinate transformation matrix between Cartesian and spherical coordinates

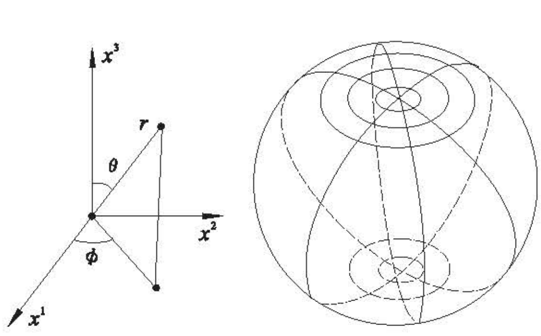

Figure 2 Diagram of curves in spherical coordinates with fixed r

3 Partial coordinate frame

Suppose $q_i(i=1,2,3)$ to be the curvilinear coordinate of point P in an OCC system within a connected domain Ω and $x_i(i=1,2,3)$ is the coordinate of P in a Cartesian coordinate system in E^3, then the Gaussian expression of the position vector $\boldsymbol{X}=OP$ from the Cartesian origin O to point P is

$$\boldsymbol{X}=(x_1,X_2,X_3) \tag{7}$$

By individually taking partial derivatives of q_1,q_2,q_3, we may obtain the coordinate transformation matrix $[\boldsymbol{T}]_{ij}$[14] from Cartesian to OCC, which is

$$[\boldsymbol{T}]_{ij}=\begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \frac{\partial x_1}{\partial q_3} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_2}{\partial q_3} \\ \frac{\partial x_3}{\partial q_1} & \frac{\partial x_3}{\partial q_2} & \frac{\partial x_3}{\partial q_3} \end{bmatrix} \tag{8}$$

Each column represents the tangent vectors of the point along the coordinate curves in the OCC denoted by $v_i(i=1,2,3)$, and

$$v_i=\left(\frac{\partial x_1}{\partial q_i}\quad\frac{\partial x_2}{\partial q_i}\quad\frac{\partial x_3}{\partial q_i}\right)^{\mathrm{T}} \tag{9}$$

Here the three tangent vectors are orthogonal to each other. Similarly, we can obtain the coordinate transformation matrix from OCC to Cartesian $[\boldsymbol{T}^{-1}]_{ij}$[15] as

$$[\boldsymbol{T}^{-1}]_{ij}=\begin{bmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} & \frac{\partial q_1}{\partial x_3} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} & \frac{\partial q_2}{\partial x_3} \\ \frac{\partial q_3}{\partial x_1} & \frac{\partial q_3}{\partial x_2} & \frac{\partial q_3}{\partial x_3} \end{bmatrix} \tag{10}$$

Each row represents the gradient (or normal vector) of $q_i(i=1,2,3)$ along the curvilinear surface at point P. Let the normal vector be v_i', then

$$v_i'=\left(\frac{\partial q_i}{\partial x_1}\quad\frac{\partial q_i}{\partial x_2}\quad\frac{\partial q_i}{\partial x_3}\right)^{\mathrm{T}} \tag{11}$$

We should point out that, in OCC, V_i and V_i' must be collinear, though not necessarily equal in length[14], which is different from a non-OCC system.

In this way, we may construct a partial orthogonal coordinate frame $V_i(i=1,2,3)$ at point P, denoted as $\{p:v_1,v_2,v_3\}$. Since the modulus and direction of this coordinate frame is a function of the point's coordinates, the frame is also called a moving coordinate frame. Thus, all curvilinear coordinates with partial frames being orthogonal are defined as OCC, and this is the content of the OCC system.

From $v_i(i=1,2,3)$, we may derive the metric tensor matrix at this point in OCC as [14]

$$[\boldsymbol{g}]_{ij}=\begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix} \tag{12}$$

Where $h_1^2=g_{ii}$, h_i is the Lame coefficient, g_{ii} is the metric tensor components or the first-class basic metric components.

And we have

$$g_{ij}=\langle \boldsymbol{v}_i,\boldsymbol{v}_j\rangle=|\boldsymbol{v}_i||\boldsymbol{v}_j|\langle \boldsymbol{e}_i,\boldsymbol{e}_j\rangle=h_ih_j\delta_j^i \tag{13}$$

Where $\langle\ ,\ \rangle$ stands for inner product of two vectors, $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ are unit vectors of $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$. and

$$\boldsymbol{v}_i=h_i\boldsymbol{e}_i \tag{14}$$

From (14), we have

$$h_i=|\boldsymbol{v}_i| \tag{15}$$

Assuming ds to be an infinitesimal arc-length element and using Einstein notations, we have

$$ds^2=g_{ij}dq_idq_j \tag{16}$$

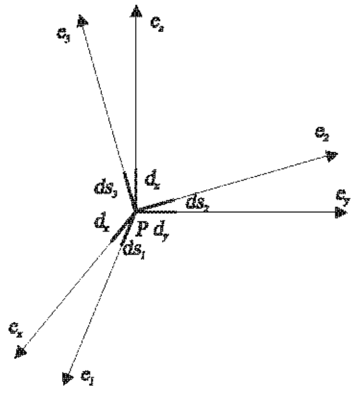

Figure 3 Infinitesimal orthogonal frame diagram at point P in Cartesian coordinates and OCC r

Equation(16) is called the first-class basic form of the metric tensor in the curvilinear coordinate system, and g_{ij} is called the metric tensor components or the first-class basic components. For OCC, $g_{ij}=0(i\neq j$, because ds^2 is a geometric quantity which is invariant with the change of coordinates. In other words, a metric tensor does not vary with coordinate transformation, but its components do.

Let ds_i be the arc differential along the coordinate curve q_i, then(16) leads to

$$ds_i = h_i dq_i \tag{17}$$

ds_i is collinear with v_i, and $ds_i(i=1,2,3)$ creates an infinitesimal orthogonal frame $\{p:ds_1,ds_2,ds_3\}$ at point *P*.

4 Derivation of a general expression

In Fig. 3, *P* is a point at the earth's surface, $\{p:ds_1, ds_2, ds_3\}$ and $\{p: dx, dy, dz\}$ are two infinitesimal frames constituted by arc differentials at point *P* in OCC and Cartesian coordinates, respectively, and $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ are unit vectors of these two frames. Let $\boldsymbol{c}$ be the transformation matrix from Cartesian to OCC, then

$$\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{pmatrix}=\boldsymbol{c}\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix} \tag{18}$$

Since $\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{pmatrix}$ and $\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix}$ are both orthogonal matrices, $\boldsymbol{c}$ is an orthogonal matrix also[4], and $\boldsymbol{c}^{-1}=\boldsymbol{c}^{\mathrm{T}}$. Considering that $\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix}$ is the unit matrix, then

$$\boldsymbol{c}=\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix} \tag{19}$$

From(18) we can obtain

$$\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix}=\boldsymbol{c}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{pmatrix} \tag{20}$$

According to paper [4], if point *P* is located at where the crust is shifted, then the infinitesimal frame $\{p: dx, dy, dz\}$ is deformed and the components of strain tensor caused by its displacement can be expressed as:

$$\begin{cases}\boldsymbol{\varepsilon}_{xx}=\dfrac{\partial u}{\partial x}\\ \boldsymbol{\varepsilon}_{xy}=\dfrac{1}{2}\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right)\\ \boldsymbol{\varepsilon}_{yy}=\dfrac{\partial v}{\partial y}\\ \boldsymbol{\varepsilon}_{zx}=\dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right)\\ \boldsymbol{\varepsilon}_{zz}=\dfrac{\partial w}{\partial z}\\ \boldsymbol{\varepsilon}_{yz}=\dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right)\end{cases} \tag{21}$$

Where $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are the components of the displacement vector along the unit vector $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$, which is a function of point *P*'s coordinates.

As mentioned above, the displacement at point *P* is a geometric quantity which does not vary in different coordinate systems, though its components and differential components do. Assuming the expression of displacement vector $\boldsymbol{u}$ in OCC to be

$$\boldsymbol{u}=\begin{pmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\\ \boldsymbol{u}_3\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{pmatrix} \tag{22}$$

$\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ being vector $\boldsymbol{u}$'s components along the unit vector $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$; the differential $\Delta\boldsymbol{u}$ has components $\Delta U_1, \Delta U_2, \Delta U_3$ along $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. Taking derivative on both sides of equation(22) gives

$$\Delta\boldsymbol{u}=\Delta\begin{pmatrix}U_1\\ U_2\\ U_3\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{pmatrix}=\Delta\begin{pmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\\ \boldsymbol{u}_3\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{pmatrix}+\begin{pmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\\ \boldsymbol{u}_3\end{pmatrix}^{\mathrm{T}}\Delta\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{pmatrix} \tag{23}$$

Substituting(18) into equation(23) leads to

$$\Delta\begin{pmatrix}U_1\\ U_2\\ U_3\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{pmatrix}=\Delta\begin{pmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\\ \boldsymbol{u}_3\end{pmatrix}^{\mathrm{T}}\left(c\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix}\right)+\begin{pmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\\ \boldsymbol{u}_3\end{pmatrix}^{\mathrm{T}}\left(\boldsymbol{\Delta}\,c\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix}+c\Delta\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix}\right) \tag{24}$$

By substituting (20) into (24) and by considering $\boldsymbol{c}\cdot\boldsymbol{c}^{\mathrm{T}}=\boldsymbol{I}$(unit matrix) and[14]:

$$\begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{25}$$

We have

$$\Delta \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}^{\mathrm{T}} = \Delta \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}^{\mathrm{T}} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}^{\mathrm{T}} (\Delta c \cdot c^{\mathrm{T}}) \tag{26}$$

In Cartesian coordinates, the transformation matrix to itself is

$$c = \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} \tag{27}$$

By substituting (27) into (26), we obtain the expression of total derivative of the displacement vector in Cartesian coordinate system as

$$\Delta \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix}^{\mathrm{T}} = \Delta \begin{pmatrix} u \\ v \\ w \end{pmatrix}^{\mathrm{T}} + \begin{pmatrix} u \\ v \\ w \end{pmatrix}^{\mathrm{T}} \left(\Delta \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} \cdot \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} \right)^{\mathrm{T}} \tag{28}$$

Substituting (25) into (28) gives

$$\Delta \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix}^{\mathrm{T}} = \Delta \begin{pmatrix} u \\ v \\ w \end{pmatrix}^{\mathrm{T}} \tag{29}$$

From paper [14], the metric tensor matrix in the Cartesian coordinates is also a unit vector, and thus the Lame coefficients are

$$h_x = h_y = h_z = \mathrm{I} \tag{30}$$

By substituting (30) to (17), we have

$$\begin{cases} \Delta s_x = h_x \Delta x = \Delta x \\ \Delta s_y = h_y \Delta y = \Delta y \\ \Delta s_z = h_z \Delta z = \Delta z \end{cases} \tag{31}$$

From formula (29) and (31) we see that equation (21) is the result of substituting specific values into the unit vector and Lame coefficients in Cartesian coordinates. Consequently, the expressions have lost Lame coefficients of metric tensor and unit vector which can characterize the Cartesian coordinates space. To restore them in this equation, let us substitute $\Delta \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix}^{\mathrm{T}}$, the total differentiation of displacement vector shown in (28), into $\Delta \begin{pmatrix} u \\ v \\ w \end{pmatrix}^{\mathrm{T}}$, and replace Δx、Δy、Δz with $\Delta s_x (h_x \Delta x)$、$\Delta s_y (h_y \Delta y)$、$\Delta s_z (h_z \Delta z)$ in equation (21).

For convenience of derivation, according to the definition of a partial derivative we omit the steps of taking the limit $\varepsilon_{xx} = \lim\limits_{\Delta s_x \to 0} \dfrac{\Delta U_x}{\Delta s_x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta U_x}{h_x \Delta x} = \dfrac{\partial U_x}{h_x \partial x}$, and directly replace the incremental "Δ" with "∂", and replace corresponding "Δ" on the right side of equation (28) with "∂" Then (21) changes into

$$\begin{cases} \varepsilon_{xx} = \dfrac{\partial U_x}{h_x \partial x}, & \varepsilon_{xy} = \dfrac{1}{2}\left(\dfrac{\partial U_x}{h_y \partial y} + \dfrac{\partial U_y}{h_x \partial x} \right) \\ \varepsilon_{yy} = \dfrac{\partial U_y}{h_y \partial y}, & \varepsilon_{xz} = \dfrac{1}{2}\left(\dfrac{\partial U_x}{h_z \partial z} + \dfrac{\partial U_z}{h_x \partial x} \right) \\ \varepsilon_{zz} = \dfrac{\partial U_z}{h_z \partial z}, & \varepsilon_{yz} = \dfrac{1}{2}\left(\dfrac{\partial U_y}{h_z \partial z} + \dfrac{\partial U_z}{h_y \partial y} \right) \end{cases} \tag{32}$$

Expression (23) shows that the strain tensor in Cartesian coordinates is a function of the unit vector and Lame coefficients. According to the principle that "Things of common nature have similarity in some aspects[15]", we can see that the commonality here is that the strain tensor in OCC, as in the Cartesian coordinates, is a function of unit vector and Lame coefficients. By using the principle of "from the specific to the common", we need only to replace the footnote x, y, z with 1, 2, 3 in equation (32) and (28), as well as u, v, w with u_1, u_2, u_3 in (28), then the frame $\{p: \mathrm{d}x, \mathrm{d}y, \mathrm{d}z\}$ in figure 3 turns into a frame $\{p: \mathrm{d}s_1, \mathrm{d}s_2, \mathrm{d}s_3\}$, finally we obtain a general expression of strain tensor in OCC as:

$$\begin{cases} \varepsilon_{ii} = \dfrac{\partial U_i}{h_i \partial q_i}, \quad i = 1,2,3 \\ \varepsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial U_i}{h_j \partial q_j} + \dfrac{\partial U_j}{h_i \partial qi} \right), \quad i \neq j, \quad i,j = 1,2,3 \end{cases} \tag{33}$$

and a rotation strain tensor expression as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{h_j \partial q_j} - \frac{\partial U_j}{h_i \partial qi} \right) \tag{34}$$

Where

$$\partial\begin{pmatrix}U_1\\U_2\\U_3\end{pmatrix}^{\mathrm{T}}=\partial\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}^{\mathrm{T}}+\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}^{\mathrm{T}}(\partial\boldsymbol{c}\cdot\boldsymbol{c}^{\mathrm{T}}),\boldsymbol{c}=\begin{pmatrix}\boldsymbol{e}_1\\\boldsymbol{e}_2\\\boldsymbol{e}_3\end{pmatrix}\tag{35}$$

and $\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}^{\mathrm{T}}(\partial\boldsymbol{c}\cdot\boldsymbol{c}^{\mathrm{T}})$ is called the correction of $\partial\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}^{\mathrm{T}}$. By these steps, the strain tensor expression in OCC is unified with the expression in Cartesian coordinates. The partial frame in OCC is orthogonal, like that in Cartesian coordinates, indicating similar characteristics. But since this partial frame is moving, so it is necessary to make a correction to adjust the differential of displacement vectors $\partial\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}^{\mathrm{T}}$ as seen in equation (35). In the case of Cartesian coordinates, this correction is zero.

By this general expression, we can derive the strain tensor expression in cylindrical and ellipsoidal coordinate systems easily.

4.1 Cylindrical coordinate system

The expression of position vector of point P in this system is:

$$\boldsymbol{X}=((R_0+r)\cos L,(R_0+r)\sin L,h)\tag{36}$$

Where R_0 is the radius of the reference cylinder, r is the normal distance from P to reference cylinder surface, L is the angle between the plane xoz and the plane determined by point P and the axis Z, h is the coordinate value of point P along Z axis.

$$\begin{cases}h_r=1, & e_r=(\cos L,\sin L,0)\\h_L=(R_0+r), & e_L=(-\sin L,\cos L,0)\\h_h=1, & e_h=(0,0,1)\end{cases}\tag{37}$$

The transformation matrix from Cartesian coordinates to cylindrical coordinates is

$$\boldsymbol{c}=\begin{pmatrix}\cos L & \sin L & 0\\-\sin L & \cos L & 0\\0 & 0 & 1\end{pmatrix}$$

$$\partial\boldsymbol{c}\cdot\boldsymbol{c}^{\mathrm{T}}=\begin{pmatrix}0 & \partial L & 0\\-\partial L & 0 & 0\\0 & 0 & 0\end{pmatrix}\tag{38}$$

$$\begin{pmatrix}\partial U_r\\\partial U_L\\\partial U_h\end{pmatrix}^{\mathrm{T}}=\begin{pmatrix}\partial u_r\\\partial u_L\\\partial u_h\end{pmatrix}^{\mathrm{T}}+\begin{pmatrix}u_r\\u_L\\u_h\end{pmatrix}^{\mathrm{T}}(\partial\boldsymbol{c}\cdot\boldsymbol{c}^{\mathrm{T}})=\begin{pmatrix}\partial u_r-u_L\partial L\\\partial u_L+u_r\partial L\\\partial u_h\end{pmatrix}\tag{39}$$

Knowing $\frac{\partial q_i}{\partial q_j}=\delta_j^i$, we can get the strain tensor in cylindrical coordinates from formula (33) as:

$$\begin{cases}\varepsilon_{rr}=\dfrac{\partial u_r}{\partial r}\\ \varepsilon_{LL}=\dfrac{\partial u_L}{(R_0+r)\partial L}+\dfrac{u_r}{R_0+r}\\ \varepsilon_{hh}=\dfrac{\partial u_h}{\partial h}\\ \varepsilon_{rL}=\dfrac{1}{2}\left(\dfrac{\partial u_r}{(R_0+r)\partial L}-\dfrac{u_L}{R_0+r}+\dfrac{\partial u_L}{\partial r}\right)\\ \varepsilon_{rh}=\dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial h}+\dfrac{\partial u_h}{\partial r}\right)\\ \varepsilon_{Lh}=\dfrac{1}{2}\left(\dfrac{\partial u_L}{\partial h}+\dfrac{\partial u_h}{(R_0+r)\partial L}\right)\end{cases}\tag{40}$$

4.2 Ellipsoidal coordinates

The expression of position vector of point P in ellipsoidal coordinates is[1,2,9]:

$$X=\left((R_1+h)\sin\theta\cos L,(R_1+h)\sin\theta\sin L,\left(\frac{b}{v}+h\right)\cos\theta\right)\tag{41}$$

Where θ, L, h are, respectively, the geodetic colatitude, longitude and elevation of point P; R_and R_2 are its principal radii of curvature of prime verticals and meridians, respectively, at the intersection of the ellipsoid surface and its normal; $v=\frac{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}{b}$, where a、b are, respectively, the long radius and short radius of the rotation ellipsoid.

$$\begin{cases} h_\theta = R_2 + h, \\ \boldsymbol{e}_\theta = (\cos\theta\cos L, \cos\theta\sin L, -\sin\theta) \\ h_L = R_1\sin\theta + h, \\ \boldsymbol{e}_L = (-\sin L, \cos L, 0) \\ h_h = 1, \\ \boldsymbol{e}_h = (\sin\theta\cos L, \sin\theta\sin L, \cos\theta) \end{cases} \tag{42}$$

The transformation matrix from the Cartesian coordinates to the ellipsoidal coordinates is:

$$\begin{aligned} \boldsymbol{c} &= \begin{pmatrix} \cos\theta\cos L & \cos\theta\sin L & -\sin\theta \\ -\sin L & \cos L & 0 \\ \sin\theta\cos L & \sin\theta\sin L & \cos\theta \end{pmatrix} \\ \partial\boldsymbol{c}\cdot\boldsymbol{c}^{\mathrm{T}} &= \begin{pmatrix} 0 & \cos\theta\partial L & -\partial\theta \\ -\cos\theta\partial L & 0 & -\sin\theta\partial L \\ \partial\theta & \sin\theta\partial L & 0 \end{pmatrix} \end{aligned} \tag{43}$$

$$\begin{cases} \partial U_\theta = \partial u_\theta - u_L\cos\theta\partial L + u_h\partial\theta \\ \partial U_L = \partial u_L + u_\theta\cos\theta\partial L + u_h\sin\theta\partial L \\ \partial U_h = \partial u_h - u_\theta\partial\theta - u_L\sin\theta\partial L \end{cases} \tag{44}$$

Since h can be ignored in comparison with the curvature radius of the earth and $\frac{\partial q_i}{\partial q_j} = \delta_j^i$, we may obtain from formula (33):

$$\begin{cases} \varepsilon_{\theta\theta} = \dfrac{1}{R_2}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_h}{R_2} \\ \varepsilon_{LL} = \dfrac{1}{R_1\sin\theta}\dfrac{\partial u_L}{\partial L} + \dfrac{\cot\theta}{R_1}u_\theta + \dfrac{u_h}{R_1} \\ \varepsilon_{hh} = \dfrac{\partial u_h}{\partial h} \\ \varepsilon_{\theta L} = \dfrac{1}{2}\left(\dfrac{1}{R_1\sin\theta}\dfrac{\partial u_\theta}{\partial L} - \dfrac{\cot\theta}{R_1}u_L + \dfrac{1}{R_2}\dfrac{\partial u_L}{\partial\theta}\right) \\ \varepsilon_{\theta h} = \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial h} - \dfrac{u_\theta}{R_2} + \dfrac{1}{R_2}\dfrac{\partial u_h}{\partial\theta}\right) \\ \varepsilon_{Lh} = \dfrac{1}{2}\left(\dfrac{\partial u_L}{\partial h} - \dfrac{u_L}{R_1} + \dfrac{1}{R_1\sin\theta}\dfrac{\partial u_h}{\partial L}\right) \end{cases} \tag{45}$$

Let $R_1 = R_2 = R$, $h = r$ formula (45) becomes the strain tensor expression in spherical coordinates. By using equation (44) we can explain why paper [18] made the assertion that " vertical displacement on sphere surface can produce positive strain in both longitudinal and latitudinal directions. Longitudinal displacement may produce positive latitudinal strain, and latitudinal displacement may produce shear strain. "

5 Concluding remarks

Expressions of strain tensor in an orthogonal curvilinear coordinate system were derived by two foreign scholars in 1981[10] and two Chinese researchers in 1992[11]. Since the results are basically similar, we only give the result of article[11] below for comparison:

$$\begin{cases} \varepsilon_{ii} = \dfrac{\partial u_i}{h_i\partial q_i} + \sum\limits_{k\neq i} u_k\dfrac{\partial \ln h_i}{h_k\partial q_k} \quad i = 1,2,3 \\ \varepsilon_{ij} = \dfrac{1}{2}\left(h_i\dfrac{\partial}{h_j\partial q_j}\left(\dfrac{u_i}{h_i}\right) + h_j\dfrac{\partial}{h_i\partial q_i}\left(\dfrac{u_j}{h_j}\right)\right) i\neq j, i,j = 1,2,3 \end{cases} \tag{46}$$

To compare with (33), we unfold (46) into a practical expression:

$$\begin{cases} \varepsilon_{11} = \dfrac{\partial u_1}{h_1\partial q_1} + \dfrac{u_2}{h_1h_2}\dfrac{\partial h_1}{\partial q_2} + \dfrac{u_3}{h_1h_3}\dfrac{\partial h_1}{\partial q_3} \\ \varepsilon_{22} = \dfrac{\partial u_2}{h_2\partial q_2} + \dfrac{u_3}{h_2h_3}\dfrac{\partial h_2}{\partial q_3} + \dfrac{u_1}{h_2h_1}\dfrac{\partial h_2}{\partial q_1} \\ \varepsilon_{33} = \dfrac{\partial u_3}{h_3\partial q_3} + \dfrac{u_1}{h_3h_1}\dfrac{\partial h_3}{\partial q_1} + \dfrac{u_2}{h_3h_2}\dfrac{\partial h_3}{\partial q_2} \\ \varepsilon_{12} = \dfrac{1}{2}\left[\dfrac{\partial u_1}{h_2\partial q_2} - \dfrac{u_1}{h_1h_2}\dfrac{\partial h_1}{\partial q_2} + \dfrac{\partial u_2}{h_1\partial q_1} - \dfrac{u_2}{h_1h_2}\dfrac{\partial h_2}{\partial q_1}\right] \\ \varepsilon_{13} = \dfrac{1}{2}\left[\dfrac{\partial u_1}{h_3\partial q_3} - \dfrac{u_1}{h_1h_3}\dfrac{\partial h_1}{\partial q_3} + \dfrac{\partial u_3}{h_1\partial q_1} - \dfrac{u_3}{h_1h_3}\dfrac{\partial h_3}{\partial q_1}\right] \\ \varepsilon_{23} = \dfrac{1}{2}\left[\dfrac{\partial u_2}{h_3\partial q_3} - \dfrac{u_2}{h_2h_3}\dfrac{\partial h_2}{\partial q_3} + \dfrac{\partial u_3}{h_2\partial q_2} - \dfrac{u_3}{h_2h_3}\dfrac{\partial h_3}{\partial q_2}\right] \end{cases} \tag{47}$$

Expressions (47) and (33) have their respective features:

1) In this study, we have proceeded from the strain tensor formula in Cartesian coordinate system, used the transformation matrix of partial unit vector frames between Cartesian coordinates and OCC, and derived a correction formula for displacement component differential in OCC. We then substituted the corrected differential and Lame coefficients into the counterparts in Cartesian coordinates, and derived a universal strain tensor expression. The critical derivation in this study is the correction for displacement vector differential (*i. e.*, the second term of equation (35)). The process is rather concise and the obtained expression has the same form as the result in Cartesian coordinates. Meanwhile, the transformation matrix of correction in (35) is a 3 × 3 order matrix consists of only cosine and sine functions of the curvi-

linear coordinate q_1 and q_2 in the curvilinear reference surface; also to take partial derivative of it is quite easy.

2) According to the definition of intrinsic geometry of a curvilinear surface, the arc length, angle, area, and volume in a flat, cylindrical, ellipsoidal or other curvilinear surface are all functions of the metric tensors in their metric space[19 – 21]. From (33) and (47) we can see that the strain tensors in OCC are all functions of their metric tensors.

3) Formula (33) and (47) are much different in the form of expression. The former is a function of h_i and $\boldsymbol{e}_i$ ($i = 1,2,3$), whereas the latter is only a function of h_i ($i = 1,2,3$). Nevertheless, they are exactly the same in essence. We can demonstrate that formula (47) can be derived from (33) on the basis of a correction of the displacement differential in (35). By stretching out the correction ∂c in (35), we get:

$$\partial c = \partial\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial e_1}{\partial q_1}\partial q_1 + \frac{\partial e_1}{\partial q_2}\partial q_2 + \frac{\partial e_1}{\partial q_3}\partial q_3 \\ \frac{\partial e_2}{\partial q_1}\partial q_1 + \frac{\partial e_2}{\partial q_2}\partial q_2 + \frac{\partial e_3}{\partial q_3}\partial q_3 \\ \frac{\partial e_3}{\partial q_1}\partial q_1 + \frac{\partial e_3}{\partial q_2}\partial q_2 + \frac{\partial e_3}{\partial q_3}\partial q_3 \end{pmatrix} \tag{48}$$

The expression of the nine partial derivatives of a unit vector in OCC with respect to the coordinates is[10, 11, 22, 23]:

$$\begin{cases} \frac{\partial e_1}{\partial q_1} = -\frac{1}{h_2}\frac{\partial h_1}{\partial q_2}e_2 - \frac{1}{h_3}\frac{\partial h_1}{\partial q_3}e_3 \\ \frac{\partial e_2}{\partial q_1} = \frac{1}{h_2}\frac{\partial h_1}{\partial q_2}e_1 \\ \frac{\partial e_1}{\partial q_2} = \frac{1}{h_1}\frac{\partial h_2}{\partial q_1}e_2 \\ \frac{\partial e_2}{\partial q_2} = -\frac{1}{h_3}\frac{\partial h_2}{\partial q_3}e_3 - \frac{1}{h_1}\frac{\partial h_2}{\partial q_1}e_1 \\ \frac{\partial e_3}{\partial q_2} = \frac{1}{h_3}\frac{\partial h_2}{\partial q_3}e_2 \\ \frac{\partial e_2}{\partial q_3} = \frac{1}{h_2}\frac{\partial h_3}{\partial q_2}e_3 \\ \frac{\partial e_3}{\partial q_3} = -\frac{1}{h_1}\frac{\partial h_3}{\partial q_1}e_1 - \frac{1}{h_2}\frac{\partial h_3}{\partial q_2}e_2 \\ \frac{\partial e_1}{\partial q_3} = \frac{1}{h_1}\frac{\partial h_3}{\partial q_1}e_3 \\ \frac{\partial e_3}{\partial q_1} = \frac{1}{h_3}\frac{\partial h_1}{\partial q_3}e_1 \end{cases} \tag{49}$$

If we substitute (49) into (48) and realize $\langle e_i, e_j \rangle = \delta_j^i$ then

$$\partial c \cdot c^{\mathrm{T}} = \begin{pmatrix} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{pmatrix} \tag{50}$$

Here

$$\begin{cases} \Omega_{12} = -\frac{1}{h_2}\frac{\partial h_1}{\partial q_2}\partial q_1 + \frac{1}{h_1}\frac{\partial h_2}{\partial q_1}\partial q_2 \\ \Omega_{13} = -\frac{1}{h_3}\frac{\partial h_1}{\partial q_3}\partial q_1 + \frac{1}{h_1}\frac{\partial h_3}{\partial q_1}\partial q_3 \\ \Omega_{23} = -\frac{1}{h_3}\frac{\partial h_2}{\partial q_3}\partial q_2 + \frac{1}{h_2}\frac{\partial h_3}{\partial q_2}\partial q_3 \end{cases} \tag{51}$$

By From (50) we may obtain the correction expression as follows

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}^{\mathrm{T}} (\partial c \cdot c^{\mathrm{T}}) = \begin{pmatrix} -\Omega_{12}u_2 - \Omega_{13}u_3 \\ \Omega_{12}u_1 - \Omega_{23}u_3 \\ \Omega_{13}u_1 + \Omega_{23}u_2 \end{pmatrix}^{\mathrm{T}} \tag{52}$$

By putting (52) into (35) and considering (51), we have

$$\partial\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \partial u_1 + \frac{u_2}{h_2}\frac{\partial h_1}{\partial q_2}\partial q_1 - \frac{u_2}{h_1}\frac{\partial h_2}{\partial q_1}\partial q_2 + \frac{u_3}{h_3}\frac{\partial h_1}{\partial q_3}\partial q_1 - \frac{u_3}{h_1}\frac{\partial h_3}{\partial q_1}\partial q_3 \\ \partial u_2 - \frac{u_1}{h_2}\frac{\partial h_1}{\partial q_2}\partial q_1 + \frac{u_1}{h_1}\frac{\partial h_2}{\partial q_1}\partial q_2 + \frac{u_3}{h_3}\frac{\partial h_2}{\partial q_3}\partial q_2 - \frac{u_3}{h_2}\frac{\partial h_3}{\partial q_2}\partial q_3 \\ \partial u_3 - \frac{u_1}{h_3}\frac{\partial h_1}{\partial q_3}\partial q_1 + \frac{u_1}{h_1}\frac{\partial h_3}{\partial q_1}\partial q_3 - \frac{u_2}{h_3}\frac{\partial h_2}{\partial q_3}\partial q_2 + \frac{u_2}{h_2}\frac{\partial h_3}{\partial q_2}\partial q_3 \end{pmatrix}^{\mathrm{T}} \tag{53}$$

By further substituting (53) into (33) while noticing that $\frac{\partial q_i}{\partial q_j} = \delta_j^i$, we find that the final result is exactly the same as (47). In (53), the sum of terms 2 to 5 on the right side of the equation is actually the contribution of the other displacement components to this displacement component's differential, which is also the expression of correction $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}^{\mathrm{T}} (\partial c \cdot c^{\mathrm{T}})$ in formula (35) denoted by Lame coefficient. It is more complicated to use formula(35) to derive strain tensor, because $\partial \boldsymbol{c} \cdot \boldsymbol{c}^{T}$ is a 3-order matrix formed by the unit vectors in OCC as row vectors and the direction cosines of curvilinear coordinates q_1 and q_2 as the variable. Thus, using formula (35) to take partial derivatives of the correction of the corresponding displacement vector differentials is naturally easier than using formula(53). For instance, in an ellipsoidal coordinate system, $h_\theta = R_2 + h$, $h_L = R_1 \sin\theta + h$, and $R_1 = \frac{c}{v}$, $R_2 = \frac{c}{v^3}$, $c = \frac{a^2}{b}$, $v = \frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}{b}$, a and b are, respectively, the long axis and short axis of the rotation ellipsoid. To solve for $\frac{\partial h_\theta}{\partial \theta}$ and $\frac{\partial h_L}{\partial \theta}$ is more complicated in this case.

Conference

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